

# Non-equilibrium thermodynamics analysis of rotating counterflow superfluid turbulence

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## Abstract

In two previous papers two evolution equations for the vortex line density  $L$ , proposed by Vinen, were generalized to rotating superfluid turbulence and compared with each other. Here, the already generalized alternative Vinen equation is extended to the case in which counterflow and rotation are not collinear. Then, the obtained equation is considered from the viewpoint of non-equilibrium thermodynamics. According with this formalism, the compatibility between this evolution equation for  $L$  and that one for the velocity of the superfluid component is studied. The compatibility condition requires the presence of a new term dependent on the anisotropy of the tangle, which indicates how the friction force depends on the rotation rate.

## 1 Introduction

Quantum turbulence is described as a chaotic motion of quantized vortices in a disordered tangle, which in some occasions shows diffusive or undulatory behavior [1]–[7]. The measurements of vortex lines are described in terms of a macroscopic average of the vortex line length per unit volume  $L$  (briefly called *vortex line density* and which has dimensions  $length^{-2}$ ).

It is well-known in literature that when one apply a rotation to a sample filled of helium II, a vortex array aligned to the rotation axis is formed. In

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the steady state  $L$  tends to  $2\Omega/\kappa$ ,  $\Omega = |\mathbf{\Omega}|$  being the modulus of the angular velocity and  $\kappa = h/m$  the quantum of vorticity ( $m$  the mass of the  $^4\text{He}$  atom and  $h$  Planck's constant,  $\kappa \simeq 9.97 \cdot 10^{-4} \text{cm}^2/\text{s}$ ).

In the experiments on thermal counterflow, instead, an almost isotropic tangle of quantized vortices is created, with  $L^{1/2} = \gamma V/\kappa$ ,  $\gamma$  being a dimensionless coefficient which depends on temperature and  $V$  the modulus of the counterflow velocity defined below. These experiments are performed by means of channel filled of helium II and heated at one end. Using the two-fluid model, since only the normal fluid component carries entropy and heat flow, its motion will be from the heated end to the opposite end. At the same time, to conserve the mass the superfluid component must counterflow towards the heater. Thus, a relative counterflow between the normal fluid and superfluid components is established, characterized by  $\mathbf{V} = \mathbf{V}_n - \mathbf{V}_s$  ( $\mathbf{V}_n$  and  $\mathbf{V}_s$  being the velocities of the normal and superfluid components averaged in a small volume  $\Lambda$ ). When counterflow velocity is higher than a critical value then quantized vortices will be created.

Neglecting the influence of the walls, an evolution equation for  $L$  under constant values of the counterflow velocity  $\mathbf{V}$  was formulated by Vinen [8]

$$\frac{dL}{dt} = \alpha V L^{3/2} - \beta \kappa L^2, \quad (1)$$

$\alpha$  and  $\beta$  being dimensionless parameters. The steady state solutions of equation (1), apart from  $L = 0$ , is  $L = (\alpha V/\beta \kappa)^2$ .

There exists another version of (1), the so-called alternative Vinen equation [9]–[10]

$$\frac{dL}{dt} = A_1 \frac{V^2}{\kappa} L - \beta \kappa L^2, \quad (2)$$

which is also admissible on dimensional grounds. The steady state solution is  $L = A_1 V^2/\beta \kappa^2$ , in agreement with the experimental results in completely developed turbulent regime.

The natural question which arises is what happens when the combined situation of rotation and counterflow exists. Supported also by experimental results [11, 12], this has addressed many recent studies [13]–[14] in this direction, some of which focusing the attention to extend Vinen's ideas to a wider range of situations [15]–[18].

One of the open problems on this arguments regards the behaviour of the turbulence when the rotation axis and the counterflow direction are neither parallel nor orthogonal, two distinguished cases already taken into account experimentally [11, 12]. The parallel case, considered by Swanson *et al.* in [11], was already investigated in Ref. [15] — where the authors

extended Vinen equation (1) to this more complicated situation — and in Ref. [18], where instead the alternative Vinen equation (2) was extended. A comparison of the theoretical studies and the experimental results was only admissible for the experiments performed by Swanson *et al.*. A motivation for the analysis of the present paper could be the behavior of a sphere filled of superfluid, which is submitted both to a rotation around its own axis (such as neutron star) and to a radial heat flux. Thus, the interaction between rotation and heat flow forming an arbitrary mutual angle arises in a natural way in these systems (see figures 2).

Because of the lack of experiments on rotating counterflow situation with  $\mathbf{V}$  and  $\mathbf{\Omega}$  forming a generic angle, some theoretical studies were made in Refs. [16, 17] extending the modified Vinen equation, proposed in [15], to this case. There, a thermodynamic analysis to determine possible coupling terms between the evolution equations of  $L$  and  $\mathbf{V}$  was performed.

In Ref. [18], the extension of Vinen equation (2) to the case of  $\mathbf{V}$  parallel to  $\mathbf{\Omega}$  was considered in order to explore whether these further arguments are enough to establish which of both equations, (1) or (2), is more suitable to describe actual experimental results. Of course, to have a satisfactory version of the macroscopic alternative Vinen equation (the same arguments are also valid for the extension of Vinen equation), the respective coefficients of all terms will have to be microscopically calculated and found to coincide with macroscopic observations. These perspectives are still far from the present abilities because of the enormous difficulties arising to model a rotating tangle. Thus, a combined effort in macroscopic and microscopic perspectives seems a reasonable and promising way to proceed.

The main aims of this paper is to extend the modified alternative Vinen equation, already proposed in [18], to include situations where the angle between  $\mathbf{V}$  and  $\mathbf{\Omega}$  is not assigned (Section 2). Furthermore, the compatibility between the generalization of the alternative Vinen equation and the evolution equation for the velocity  $\mathbf{V}_s$  is studied from the non-equilibrium thermodynamics viewpoint (Section 3). The new equations (equations (6) and (23)) model a class of physical phenomena where very few experiments are available. However, the model appears to be confirmed by its ability to describe related physical phenomena as shown in the following sections. It is to note that the results of the present paper need the support of the experiments to be confirmed and improved. In particular, the model could be applied to the neutron stars which are rotating compact self-gravitating objects whose bulk consists of a superfluid neutron liquid at the huge density, with a very small concentration of protons and electrons.

## 2 Contribution of rotation to the evolution equation for the vortex line density: general case

In Ref. [11], Swanson *et al.* and, in Ref. [12], Yarmchuck and Glaberson applied a heat flux to a rotating sample showing a complex interaction between processes of formation and destruction of vortices. Particularly, in the experiment performed by Swanson *et al.* ( $\mathbf{\Omega}$  and  $\mathbf{V}$  collinear), they observed that the effects of  $V$  and  $\Omega$  are not additive: in fact, for sufficiently high values of  $\Omega$ , when the influence of the walls can be neglected, the total vortex line density is lower than  $L_R + L_H$ ,  $L_R$  and  $L_H$  being the values of  $L$  in steady rotation and in steady counterflow superfluid turbulence, respectively, given by

$$L = L_R = \frac{2\Omega}{\kappa}, \quad L_H = \gamma^2 \frac{V^2}{\kappa^2}, \quad (3)$$

and the deviation increases with  $V$  and  $\Omega$ . Therefore, the rotation facilitates the vortex formation, in the absence or for small counterflow velocities, but it hinders their lengthening for high values of  $V$  and  $\Omega$ . Furthermore, they observed that there are two critical counterflow-rotation velocities  $V_{c1}$  and  $V_{c2}$ , proportional to  $\Omega^{1/2}$  ( $V_{c1} = C_1\sqrt{\Omega}$ ,  $V_{c2} = C_2\sqrt{\Omega}$ , with  $C_1 = 0.053 \text{ cm sec}^{-1/2}$ ,  $C_2 = 0.118 \text{ cm sec}^{-1/2}$ ). For  $V \leq V_{c2}$  the length  $L$  per unit volume of the vortex lines is independent on  $V$  and agrees with the expression (3a), but with a slightly different proportionality constant in the range  $V_{c1} \leq V \leq V_{c2}$ . Finally, for  $V \geq V_{c2}$ ,  $L$  increases and becomes proportional to  $V^2$  at high values of  $V$ .

The experiments performed, instead, by Yarmchuck and Glaberson in [12] refer to an array of vortices, caused by a rotation  $\Omega$ , which suffer the influence of an orthogonal counterflow. They measure the gradients of temperatures and chemical potential which confirms the presence of two critical velocities for the counterflow velocity, the first corresponding to a depinning of the vortices from the walls and the second one corresponding to the transition to the turbulent state. The critical velocities are proportional to  $\sqrt{\Omega}$  as pointed out by Swanson *et al.* in their experiments.

As mentioned in the Introduction, in the regime of high rotation ( $0.2 \text{ Hz} \leq \Omega/2\pi \leq 1.0 \text{ Hz}$  and  $0 \leq V^2 \leq 0.2 \text{ cm}^2/\text{s}^2$ ) and  $\mathbf{\Omega}$  parallel to  $\mathbf{V}$ , equation (1) has been generalized into [15]

$$\frac{dL}{dt} = -\beta\kappa L^2 + \alpha_1 \left[ L^{1/2} - m_1 \frac{\sqrt{\Omega}}{\sqrt{\kappa}} \right] VL + \beta_2 \left[ L^{1/2} - m_2 \frac{\sqrt{\Omega}}{\sqrt{\kappa}} \right] \sqrt{\kappa\Omega} L, \quad (4)$$

where  $m_1 = \beta_4/\alpha_1$  and  $m_2 = \beta_1/\beta_2$ , and  $\beta$ ,  $\alpha_1$ ,  $\beta_1$ ,  $\beta_2$  and  $\beta_4$  are coefficients depending on the polarization of the tangle, which was supposed to be a function of  $\Omega$  and  $V$ . As observed in [15], equation (4) describes some of the most relevant effects observed in the experiments of Ref. [11].

Since experimental results in pure counterflow are also described by the alternative Vinen equation, in [18] a generalization of it to rotating counterflow was proposed following the main lines of Ref. [15]. This generalization was carried out assuming that the counterflow velocity and the rotating axis were parallel one to each other in order to compare theoretical results with experiments by Swanson *et al.*. This equation assumed the following form

$$\frac{dL}{dt} = -\beta\kappa L^2 + A_1 \left[ L - \nu_1 \frac{\Omega}{\kappa} \right] \frac{V^2}{\kappa} + B_1 \left[ L - \nu_2 \frac{\Omega}{\kappa} \right] \Omega, \quad (5)$$

and it is valid only for  $V$  collinear to  $\Omega$ . In Eq. (5) coefficients  $\beta$ ,  $A_1$ ,  $\nu_1$ ,  $B_1$  and  $\nu_2$  were supposed to depend on the polarization of the tangle, i.e. on  $\Omega$  and  $V$ .

Now, following the general line of the paper [17] equation (5) is written to include the general situation in which vectors  $\Omega$  and  $\mathbf{V}$  are not necessarily collinear. Looking at the terms of equation (5), the only one which needs experimental examination is that proportional to  $\Omega V^2$ , because in the general case it could depend not only on the absolute values of the two vectors  $\Omega$  and  $\mathbf{V}$  but also on the angle between them. A mathematically consistent version of (5) including both alternatives is

$$\begin{aligned} \frac{dL}{dt} = & -\beta\kappa L^2 + L \left[ \frac{A_1}{\kappa} \mathbf{V} \cdot \mathbf{U} \cdot \mathbf{V} + B_1 \hat{\Omega} \cdot \mathbf{U} \cdot \Omega \right] \\ & - \left[ \frac{A_1 V \nu_1}{\kappa^2} \Omega \cdot \left( a_1 \hat{\mathbf{V}} \hat{\Omega} + a_2 \hat{\Omega} \hat{\mathbf{V}} \right) \cdot \mathbf{V} + \frac{B_1 \nu_2}{\kappa} \Omega \cdot \mathbf{U} \cdot \Omega \right], \quad (6) \end{aligned}$$

where  $\mathbf{U}$  is the second order unit tensor,  $a_1$  and  $a_2$  are two coefficients such that  $a_1 + a_2 = 1$ ,  $\hat{\mathbf{V}} \hat{\Omega}$  and  $\hat{\Omega} \hat{\mathbf{V}}$  are diadic products between  $\mathbf{V}$  and  $\Omega$  such that the first depends only on the absolute values of two vectors  $V$ ,  $\Omega$ , whereas the second one depends also on their angle,  $\cos^2(\mathbf{V}\Omega)$ .

As pointed out above, equation (6) includes different situations and it becomes equation (5) when one considers a rotating cylinder heated from the lower basis or the higher basis. A physically interesting situation arises when two vectors are not parallel, for instance when one consider a rotating cylinder heated from the wall, which means that  $\mathbf{V}$  and  $\Omega$  are perpendicular as in experiment performed in Ref. [12] (these experiments will be discussed in the Conclusions). In the general case, it is important to establish which

term,  $a_1$  and  $a_2$ , has to be included into equation (6). Since up to now we are not able to answer to this question, both coefficients  $a_1$  and  $a_2$  have to be considered.

The stationary solutions of equation (6) are obtained requiring  $dL/dt = 0$ , that is

$$-\beta\kappa L^2 + L \left( \frac{A_1}{\kappa} V^2 + B_1 \Omega \right) - \left[ \frac{A_1 V^2 \Omega \nu_1}{\kappa^2} \left( a_1 + a_2 \cos^2(\hat{\Omega} \hat{\mathbf{V}}) \right) + \frac{B_1 \nu_2}{\kappa} \Omega^2 \right] = 0. \quad (7)$$

The case  $\Omega$  parallel to  $\mathbf{V}$ , that is the stationary solutions of equation (7) with  $\cos^2(\hat{\Omega} \hat{\mathbf{V}}) = 1$  and  $a_1 + a_2 = 1$  was investigated in [18]. The main results of the paper [18] are reported in order to extend them to the general case dealt with in this paper.

In the steady state ( $L$  and  $\mathbf{V}$  constant), under the constraint

$$\nu_2 = \nu_1 - \nu_1^2 \frac{\beta}{B_1}, \quad (8)$$

the stationary solutions of equation (5) can be written

$$L = L_1^{\parallel} = \nu_1 \frac{\Omega}{\kappa}, \quad (9)$$

$$L = L_2^{\parallel} = \frac{A_1}{\beta} \frac{V^2}{\kappa^2} + \left( \frac{B_1}{\beta} - \nu_1 \right) \frac{\Omega}{\kappa}. \quad (10)$$

Solutions (9) and (10) are two families of straight lines in the plane  $(V^2, L)$ , as plotted in Fig. 1: the first of them (equation (9)) parallel to the  $V^2$  axis and the second one (equation (10)) with a slope independent of  $\Omega$ . Applying the stability analysis to these solutions, solution (9) is stable if  $V$  is lower than

$$V_{c2}^2 = \frac{\beta}{A_1} \frac{\nu_1^2 - 2\nu_1\nu_2}{\nu_1 - \nu_2} \Omega \kappa, \quad (11)$$

while, for  $V$  higher than  $V_{c2}$  the solution (10) is stable. The obtained value of the critical velocity (11) corresponds to the interception of the two straight lines (9) and (10) in Fig.1. It also corresponds to the second critical counterflow-rotation velocity observed in the experiments of Ref. [11]. As we see, this critical velocity scales as  $\sqrt{\Omega}$ , in agreement with experimental observations.

To describe the existence of the first critical velocity  $V_{c1}$  in the experiments, and also the small step in  $L$ , in Ref. [18]  $\nu_1$  was assumed dependent on  $\Omega$  and  $V$  as

$$\nu_1 = A \left\{ 1 - B \tanh \left[ N' \left( \frac{k\Omega}{V^2} - C \right) \right] \right\}. \quad (12)$$

There,  $A = 2.018$ ,  $B = 0.0089$  and  $C = 0.355$  were found in such a way that for  $V^2 \ll V_{c1}^2 = \frac{1}{C}k\Omega$ , it results  $\nu_1 \simeq A - BA = 2$  and for  $V^2 \gg V_{c1}^2$ ,  $\nu_1 = A + BA = \nu^{max} = 2.036$ , while the constant  $C$  is related to the critical value  $V_{c1}$  of the counterflow velocity by  $V_{c1}^2 = \frac{1}{C}\kappa\Omega$ , whose experimental value is  $V_{c1} = 0.053\sqrt{\Omega}$  cm sec $^{-1/2}$ . In (12),  $N'$  is a phenomenological coefficient characterizing the rate of growth of  $L$  near  $V_{c1}$  and for it the value  $N' = 22$  proposed by Tsubota *et al.* in Ref. [14] was assumed.

Now, let us investigate on the stationary solutions of equation (6). Thus, assuming that relation (8) is still valid when one substitutes  $\nu_1$  with  $\tilde{\nu}_1 = \nu_1(a_1 + a_2 \cos^2(\hat{\Omega}\hat{\mathbf{V}}))$ , that is

$$\nu_2 = \tilde{\nu}_1 - \tilde{\nu}_1^2 \frac{\beta}{B_1}, \quad (13)$$

then (7) has the following stationary solutions

$$L = L_1 = \tilde{\nu}_1 \frac{\Omega}{\kappa} = (a_1 + a_2 \cos^2(\hat{\Omega}\hat{\mathbf{V}})) \frac{\Omega}{\kappa}, \quad (14)$$

$$L = L_2 = \frac{A_1}{\beta} \frac{V^2}{\kappa^2} + \left( \frac{B_1}{\beta} - \tilde{\nu}_1 \right) \frac{\Omega}{\kappa} = \frac{A_1}{\beta} \frac{V^2}{\kappa^2} + \left( \frac{B_1}{\beta} - (a_1 + a_2 \cos^2(\hat{\Omega}\hat{\mathbf{V}})) \right) \frac{\Omega}{\kappa}. \quad (15)$$

The stability analysis furnishes the same conclusions of the parallel case, namely  $L_2$  is stable for  $V > V_{c2}$ ,  $V_{c2}$  being the second critical velocity defined by

$$V_{c2}^2 = \frac{\beta}{A_1} \frac{\tilde{\nu}_1^2 - 2\tilde{\nu}_1\nu_2}{\tilde{\nu}_1 - \nu_2} \Omega \kappa = \frac{2\tilde{\nu}_1\beta - B_1}{A_1} \Omega \kappa \quad (16)$$

and depending on the angle between  $\hat{\Omega}$  and  $\hat{\mathbf{V}}$  through  $\tilde{\nu}_1$ . Looking at the first solution  $L_1$  one notes that

$$a_1 \leq (a_1 + a_2 \cos^2(\hat{\Omega}\hat{\mathbf{V}})) \leq (a_1 + a_2) = 1$$

which means

$$\tilde{\nu}_1|_{V \perp \Omega} \leq \tilde{\nu}_1 \leq \tilde{\nu}_1^{\parallel} = \nu_1,$$

$\nu_1$  being the coefficient of the stationary solution  $L_1$  in (9) and plotted in Fig. 1. An equivalent relation could also be written for  $L_2$ , but, differently to  $L_1$ , coefficients  $A_1$  and  $B_1$  could depend on the anisotropy of the tangle, namely on the angle between  $\mathbf{V}$  and  $\Omega$ .

In Ref. [18], the dimensionless quantities appearing in the evolution equation (5) are determined by comparison with the experimental data in the steady state, obtaining

$$\frac{A_1}{\beta} = 0.0125, \quad \frac{B_1}{\beta} = 3.90, \quad \nu_1 = 2.036, \quad \nu_2 = 0.97. \quad (17)$$

Instead, in the case of pure counterflow  $A_1/\beta = 0.156$ , which confirms the dependence of the coefficients on the anisotropy of the tangle [18]. By (17), solutions (9) and (10) become

$$L_1^\parallel = 2.036 \frac{\Omega}{\kappa}, \quad \text{and} \quad L_2^\parallel = 0.0125 \frac{V^2}{\kappa^2} + 1.86 \frac{\Omega}{\kappa}. \quad (18)$$

These solutions are plotted in Fig. 1, where the agreement with the experimental data is shown.

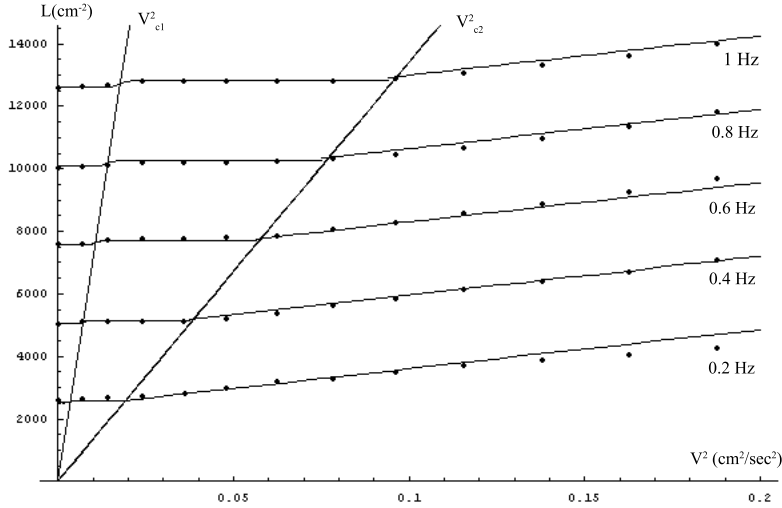


Figure 1: Comparison of the solutions (18) (continuous line) with the experimental data by Swanson *et al.* [11]. Figure from Ref. [18].

An interesting problem is to establish which equation, either the one based on the usual Vinen equation or the other one based on the alternative Vinen equation, fits better the experimental data obtained by Swanson, Barenghi and Donnelly [11]. This comparison was made in Ref. [18]: the two stationary solutions in the range  $V < V_{c2}$  represent the same straight line in the plane  $(L, V^2)$ , whereas in the range  $V^2 > V_{c2}^2$  it is not the case. From the direct comparison with the experimental data — by finding the errors between the stationary solutions of the two models and the corresponding experimental values — in Ref. [18] the following statement was established: the stationary solution of the alternative Vinen equation approaches better the experimental data (for  $V^2 > V_{c2}^2$ ) than that of the usual Vinen equation.

In the same paper the unsteady behavior of the solutions of these equations was also explored. Both equations exhibit remarkable differences, such



as the solutions of the alternative Vinen tend much faster to their steady-state values, and this difference depends on the value of the counterflow velocity. Even though detailed experimental data on this unsteady behavior lacks, according to Swanson *et al.* the time required to reach the steady state is less than 10 minutes — when the counterflow velocity  $V$  is slightly above the critical velocity  $V_{c2}$  and it increases between two consecutive experimental values (see pag. 191, Ref. [11]). According to the results of Ref. [18], the temporal scale of the solution of the usual Vinen equation was closer to the observations than the temporal scale corresponding to the alternative equation, which tends too fast to the final result. Thus, it seems that the usual equation is preferable on these grounds.

### 3 Thermodynamic analysis

In this Section the general lines of Refs. [10, 16, 17] are followed to determine an evolution equation for  $\mathbf{V}_s$  which is consistent with equation (6). This situation is of special interest when  $\mathbf{V}_s$  is not kept constant, as it is usually assumed in (1) or (2) — or in (4) and (5). This more general model may be useful to study the dynamic behavior of the vortex tangle, when the counterflow velocity  $\mathbf{V}$  varies in a greater range than that considered in Ref. [18]. According to the formalism of nonequilibrium thermodynamics, evolution equations for  $\mathbf{V}_s$  and  $L$  can be obtained by writing  $d\mathbf{V}_s/dt$  and  $dL/dt$  in terms of their conjugate forces  $-\rho_s\mathbf{V}$  and  $\epsilon_V$ . This is motivated also by the paper of Nemirowskii *et al.* [10], where for the entropy density  $s$  of the superfluid in the presence of vortex lines they considered the differential form

$$T \frac{ds}{dt} = -\rho_s \mathbf{V} \cdot \frac{d\mathbf{V}_s}{dt} + \epsilon_V \frac{dL}{dt}, \quad (19)$$

where

$$-\rho_s \mathbf{V} \equiv \frac{\partial u}{\partial \mathbf{V}_s}, \quad \epsilon_V \equiv \frac{\partial u}{\partial L} = \frac{\rho_s \kappa^2}{4\pi} \ln \left[ \frac{1}{a_0 L^{1/2}} \right], \quad (20)$$

being  $u$  the internal energy density and  $\epsilon_V$  the contribution to the internal energy per unit length of the vortex line ( $a_0$  is the dimension of the vortex core, which is very small, of the order of one Å) [7].

First, recalling that in the presence of pure rotation (which produces an ordered array of vortex lines parallel to the rotation axis) the evolution equation for  $\mathbf{V}_s$  is [7, 17, 19]

$$\frac{d\mathbf{V}_s}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V}_s + \mathbf{i}_0 = -L \frac{\rho_n}{\rho} \frac{B}{\tilde{\alpha}} \left( \hat{\boldsymbol{\Omega}} \times \hat{\boldsymbol{\Omega}} \times \mathbf{V} + \frac{B'}{B} \hat{\boldsymbol{\Omega}} \times \mathbf{V} \right), \quad (21)$$

where  $\tilde{\alpha} = 2/\kappa$ ,  $\mathbf{i}_0$  is the inertial force,  $B$  and  $B'$  are the Hall-Vinen dimensionless coefficients describing the interaction between the normal fluid and the vortex lines [19].

In the presence of pure counterflow, the evolution equation for  $\mathbf{V}_s$  can be written [10]

$$\frac{d\mathbf{V}_s}{dt} = \frac{\rho_n}{\rho} AL\mathbf{V}, \quad (22)$$

where  $A = 2B/3\tilde{\alpha}$  [7]. Since in the steady state  $L$  is proportional to  $V^2$ , the friction force on  $\mathbf{V}_s$  (which is opposite to  $\mathbf{V}$ ), will be proportional to  $V^3$ , a result known as Gorter-Mellink law [7, 2, 10]. Note, however, that this  $V^3$  dependence is valid only at the steady state, whereas (22) may be also applied in unsteady situations, in absence of rotation. Here we will be interested in the form of the friction force in unsteady situations including rotation and, in more general terms, in the general form of the evolution equation for  $\mathbf{V}_s$ .

In the simultaneous presence of counterflow and rotation, terms as those in the right-hand side of (21) and (22) must be included in the evolution equation for  $\mathbf{V}_s$ , as well as an additional contribution  $\mathbf{F}_{\text{coupl}} = \mathbf{F}_c(\mathbf{V}, \boldsymbol{\Omega})$  due to couplings between counterflow, rotation and superfluid velocity. The coupling term is found similarly to the Onsager's formalism of non-equilibrium thermodynamics [10, 16, 17]

$$\frac{d\mathbf{V}_s}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V}_s + \mathbf{i}_0 = \frac{\rho_n}{\rho} L \frac{B}{\tilde{\alpha}} \boldsymbol{\Pi} \cdot \mathbf{V} + \mathbf{F}_c(\mathbf{V}, \boldsymbol{\Omega}), \quad (23)$$

where tensor

$$\boldsymbol{\Pi} \equiv (1 - b) \frac{2}{3} \mathbf{U} + b \left( \mathbf{U} - \hat{\boldsymbol{\Omega}} \hat{\boldsymbol{\Omega}} \right) + b' \frac{B'}{B} \mathbf{W} \cdot \hat{\boldsymbol{\Omega}} \quad (24)$$

was introduced and studied in Ref. [20]. In Refs. [16, 17], for the sake of simplicity,  $b = b'$  was chosen. Here,  $b$  and  $b'$  are numerical parameters related to the anisotropy and polarization of vortex tangle, describing the relative weight of the array of vortex lines parallel to  $\boldsymbol{\Omega}$  and the isotropic tangle:  $b = b' = 0$  means an isotropic tangle corresponding to pure counterflow and  $b = b' = 1$  means an ordered array corresponding to pure rotation. The microscopic physical interpretation of coefficients  $b$  and  $b'$  was made in Ref. [20]. In particular  $b$  is linked to the anisotropy of the tangle and  $b'$  to the polarity.

Our aim here is to include equations (6) and (23) into a common thermodynamic framework, and study possible couplings between them. But,

because of the origin of the term  $\Omega V^2$  in equation (5) is unknown, as pointed out in Section 2, here special emphasis is given to the two opposite cases:  $a_1 = 1$  and  $a_2 = 0$ , and  $a_1 = 0$  and  $a_2 = 1$ . Of course, an intermediate situation where both coefficients are non zero is not excluded and could be an interesting physical situation which will not be considered for the sake of simplicity. Future experiments could urge us to spend further efforts on these intermediate cases.

### 3.1 $\Omega V^2$ interpreted as $V\boldsymbol{\Omega} \cdot \hat{\mathbf{V}}\hat{\boldsymbol{\Omega}} \cdot \mathbf{V}$ ( $a_1 = 1$ and $a_2 = 0$ )

In this subsection, term  $\Omega V^2$  is assumed to come from  $V\boldsymbol{\Omega} \cdot \hat{\mathbf{V}}\hat{\boldsymbol{\Omega}} \cdot \mathbf{V}$  which depends only on the absolute values of the two fields. In this spirit, equation (6) for  $L$  with  $a_1 = 1$  and  $a_2 = 0$  is written in the second line of the following system (25). The second part of the evolution equation for  $\mathbf{V}_s$  is built up by means of the Onsager-Casimir reciprocity relation, as in Refs. [16, 17]. The result is

$$\left[ \frac{D\mathbf{V}_s}{dt} \right] = \begin{bmatrix} -\frac{\rho_n}{\rho\rho_s} \frac{B}{\tilde{\alpha}} L \boldsymbol{\Pi} & \frac{1}{\rho_s} \frac{A_1}{\kappa} \left[ L - \nu_1 \frac{\Omega}{\kappa} \right] \mathbf{V} \\ -\frac{1}{\rho_s} \frac{A_1}{\kappa} \left[ L - \nu_1 \frac{\Omega}{\kappa} \right] \mathbf{V} & -\frac{1}{\epsilon_V} \left[ \beta \kappa L^2 - B_1 \left( L - \nu_2 \frac{\Omega}{\kappa} \right) \Omega \right] \end{bmatrix} \begin{bmatrix} -\rho_s \mathbf{V} \\ \epsilon_V \end{bmatrix}, \quad (25)$$

where

$$\frac{D\mathbf{V}_s}{dt} = \frac{d\mathbf{V}_s}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V}_s + \mathbf{i}_0. \quad (26)$$

Therefore, in the presence of counterflow and rotation the equation for  $d\mathbf{V}_s/dt$  is

$$\frac{d\mathbf{V}_s}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V}_s + \mathbf{i}_0 = L \frac{\rho_n}{\rho} \frac{B}{\tilde{\alpha}} \boldsymbol{\Pi} \cdot \mathbf{V} + \frac{\epsilon_V}{\rho_s} \frac{A_1}{\kappa} \left[ L - \nu_1 \frac{\Omega}{\kappa} \right] \mathbf{V}. \quad (27)$$

From a comparison with (23) the following expression for the coupling friction force is obtained

$$\mathbf{F}_{coupl} = \frac{\epsilon_V}{\rho_s} \frac{A_1}{\kappa} \left[ L - \nu_1 \frac{\Omega}{\kappa} \right] \mathbf{V}. \quad (28)$$

This term is proportional to  $\mathbf{V}$ , and is therefore a friction term (recall that  $\mathbf{V}$  is opposite to  $\mathbf{V}_s$ ). Note the presence of a negative term proportional to  $\Omega$  and independent on  $L$ , which means that the rotation of the vortex tangle reduces the friction. This may be interpreted as a consequence of the alignment of the vortices to the rotation axis due to  $\boldsymbol{\Omega}$ , in such a way that they have less resistance to the flow (recall that the friction is due to

vortices orthogonal to the counterflow velocity, but not to those parallel to it).

Substituting (14) and (15) in the off-diagonal term in the matrix in (25), one obtains the following expression for the coupling force

$$\mathbf{F}_{coupl} = \frac{\epsilon_V}{\rho_s} \frac{A_1}{\kappa} \left[ L_1 - \nu_1 \frac{\Omega}{\kappa} \right] \mathbf{V} \equiv 0, \quad \text{for } V < V_{c2}, \quad (29)$$

$$\mathbf{F}_{coupl} = \frac{\epsilon_V}{\rho_s} \frac{A_1}{\kappa} \left[ L_2 - \nu_1 \frac{\Omega}{\kappa} \right] \mathbf{V} \equiv \frac{\epsilon_V}{\rho_s} \frac{A_1^2}{\beta \kappa^3} (V^2 - V_{c2}^2) \mathbf{V}, \quad \text{for } V > V_{c2}. \quad (30)$$

As a consequence, in a steady situation (and also in almost steady state) for  $V < V_{c2}$  the coupling force is absent (as in pure rotation) while, for  $V > V_{c2}$  — when the array of rectilinear vortex lines becomes a disordered tangle — the additional term (30) proportional to  $\mathbf{V}$  appears. Thus,  $V_{c2}$  indicates the threshold not only of the vortex line dynamics but also of the friction acting on the velocity  $\mathbf{V}_s$  itself.

### 3.2 $\Omega V^2$ interpreted as $V \boldsymbol{\Omega} \cdot \hat{\boldsymbol{\Omega}} \hat{\mathbf{V}} \cdot \mathbf{V}$ ( $a_1 = 0$ and $a_2 = 1$ )

Here the same lines of the previous subsection are applied to the opposite case  $a_1 = 0$  and  $a_2 = 1$ , which means that term  $\Omega V^2$  comes from  $V \boldsymbol{\Omega} \cdot \hat{\boldsymbol{\Omega}} \hat{\mathbf{V}} \cdot \mathbf{V}$ , depending also on the angle between the two vectors. So, writing again equation (6) with  $a_1 = 0$  and  $a_2 = 1$  in the second line of the following system (31), the second part of the evolution equation for  $\mathbf{V}_s$  is built up by means of the Onsager-Casimir reciprocity relation

$$\begin{bmatrix} \frac{D\mathbf{V}_s}{dt} \\ \frac{dL}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{\rho_n}{\rho \rho_s} \frac{B}{\tilde{\alpha}} L \boldsymbol{\Pi} & \frac{1}{\rho_s} \frac{A_1}{\kappa} \left[ L - \tilde{L}_1 \right] \mathbf{V} \\ -\frac{1}{\rho_s} \frac{A_1}{\kappa} \left[ L - \tilde{L}_1 \right] \mathbf{V} & -\frac{1}{\epsilon_V} \left[ \beta \kappa L^2 - B_1 \left( L - \nu_2 \frac{\Omega}{\kappa} \right) \Omega \right] \end{bmatrix} \begin{bmatrix} -\rho_s \mathbf{V} \\ \epsilon_V \end{bmatrix}, \quad (31)$$

where  $\tilde{L}_1 = \nu_1 \frac{\Omega}{\kappa} \cos^2(\hat{\boldsymbol{\Omega}} \hat{\mathbf{V}})$ . In this case, instead, the equation for  $d\mathbf{V}_s/dt$  in the presence of counterflow and rotation is

$$\frac{d\mathbf{V}_s}{dt} + 2\boldsymbol{\Omega} \times \mathbf{V}_s + \mathbf{i}_0 = L \frac{\rho_n}{\rho} \frac{B}{\tilde{\alpha}} \boldsymbol{\Pi} \cdot \mathbf{V} + \frac{\epsilon_V}{\rho_s} \frac{A_1}{\kappa} \left[ L - \nu_1 \frac{\Omega}{\kappa} \cos^2(\hat{\boldsymbol{\Omega}} \hat{\mathbf{V}}) \right] \mathbf{V}. \quad (32)$$

From a comparison with (23), the following expression for the coupling friction force is obtained

$$\mathbf{F}_{coupl} = \frac{\epsilon_V}{\rho_s} \frac{A_1}{\kappa} \left[ L - \nu_1 \frac{\Omega}{\kappa} \cos^2(\hat{\boldsymbol{\Omega}} \hat{\mathbf{V}}) \right] \mathbf{V}. \quad (33)$$

Note that it is equal to (28) unless  $\cos^2(\hat{\Omega}\hat{\mathbf{V}})$ . So, all the observations below relation (28) are still valid. Note also that the negative term, proportional to  $\Omega$  and independent on  $L$ , depends explicitly on the angle between  $\mathbf{V}$  and  $\Omega$  and it is maximum for  $\mathbf{V}$  parallel to  $\Omega$  whereas it is zero for  $\mathbf{V}$  orthogonal to  $\Omega$ . The friction term has always the same direction of counterflow velocity  $\mathbf{V}$ , and in the situation  $\mathbf{V} \perp \Omega$  depends only on  $V$  and  $L$  but not on the angular velocity  $\Omega$ .

By using expressions (14) and (15), the coupling term (33) in the steady state becomes

$$\mathbf{F}_{coupl} = \frac{\epsilon_V}{\rho_s} \frac{A_1}{\kappa} \left[ L_1 - \nu_1 \frac{\Omega}{\kappa} \cos^2(\hat{\Omega}\hat{\mathbf{V}}) \right] \mathbf{V} \equiv 0, \quad \text{for } V < V_{c2}, \quad (34)$$

$$\mathbf{F}_{coupl} = \frac{\epsilon_V}{\rho_s} \frac{A_1}{\kappa} \left[ L_2 - \nu_1 \frac{\Omega}{\kappa} \cos^2(\hat{\Omega}\hat{\mathbf{V}}) \right] \mathbf{V} \equiv \frac{\epsilon_V}{\rho_s} \frac{A_1^2}{\beta \kappa^3} (V^2 - V_{c2}^2) \mathbf{V}, \quad \text{for } V > V_{c2}. \quad (35)$$

Here, the conclusions are the same pointed out below relations (29) and (30), and, therefore, also in this case  $V_{c2}$  indicates the threshold of the friction force acting on the velocity  $\mathbf{V}_s$  itself.

## 4 Conclusions

The possibility of at least two reasonable evolution equations for the vortex line density  $L$  in superfluid turbulence was known since the early days in which Vinen proposed them. This equation was generalized to the combined situation of counterflow and rotation, with  $\mathbf{V}$  parallel to  $\Omega$ , into (4) and (5), which was compared with the experimental data given by Swanson *et al.*. In [17] Vinen equation (4) was extended to include the case  $\mathbf{V}$  and  $\Omega$  with different directions. Here, following the same ideas of that paper, an extension of the modified alternative Vinen equation (5) has been proposed in (6). So, this equation is valid when  $\mathbf{V}$  and  $\Omega$  have indefinite directions, becoming the evolution equation (5) when these vectors are parallel. The stationary solutions of this extended equation differ from that of parallel case for the coefficient  $\tilde{\nu}$  instead of  $\nu$ . So, from the theoretical point of view one can establish that an higher angle between  $\Omega$  and  $\mathbf{V}$  implies a lower value for the steady state  $L_1$ , unless future experiments will prove that  $\tilde{\nu} \equiv \nu$ .

The case  $\mathbf{V}$  orthogonal to  $\Omega$  was experimentally carried out by Yarmchuck and Glaberson in Ref.[12]. They arranged a pair of horizontal parallel glass plates forming a closed channel of rectangular cross-section closed at one end with a heater nearby, and open at the other end to the liquid helium

bath. The channel was of large aspect ratio: the height being 0.5 mm, the width 1.4 cm, and the length 5.5 cm. The channel is rotated around its vertical axis, orthogonal to the direction of the heat flux, in such a way that the counterflow velocity  $\mathbf{V}$  is orthogonal to angular velocity  $\mathbf{\Omega}$ . The authors check temperature and chemical gradients as functions of heater power and rotation speed. The results were linear regimes, in which the temperature and chemical gradients increase when rotating velocity is increased, and a critical heater power  $q_{c2}$  (corresponding a critical counterflow velocity  $V_{c2}$ ), which increase as the rotation speed increases, becoming proportional to  $\sqrt{\Omega}$ .

These results points out that coefficient  $a_1$  in equation (6) is not zero. For, in the situation of the above experiment  $\mathbf{V}$  is orthogonal to  $\mathbf{\Omega}$ , namely the term proportional to  $a_2$  is zero because  $\cos^2(\hat{\mathbf{V}}\hat{\mathbf{\Omega}}) = 0$ . On the other side, experimental results confirm the presence of the term proportional to  $V^2\Omega$ . But, to establish whether the coefficient  $a_2$  is equal to zero, further experiments are required.

As mentioned in the Introduction of this paper, an interesting application of these arguments could be a rotating spherical container (like a star) filled by superfluid helium which suffers the presence of radial heat flow (see figure 2). Since irradiation comes from the center of the star, the heat flow will be proportional to  $r^{-2}$ ,  $r$  being the distance from the center of the star. Because of the proportionality between heat flow and counterflow velocity, this dependence on  $r$  requires the existence of a critical radius  $r_2$  (corresponding to the second critical velocity  $V_{c2}$ ) which marks the turbulent region ( $r < r_2$ ). Of course, the critical radius  $r_2$  depends on the angular velocity  $\Omega$  from the relation (16). An interesting question which arises could be whether  $r_2$  depends on the angle  $\theta$  between  $\mathbf{V}$  and  $\mathbf{\Omega}$ , or similarly whether  $a_2$  is zero or not. Because of the lack of experimental evidence any conclusion on the presence of the coefficient  $a_2$  is not reached. However, the case  $a_2 = 0$  would require a critical radius free from  $\theta$  (see relation (16)) and a spherical turbulent region of radius  $r_2$  as showed in figure 2a. The case  $a_2 \neq 0$ , instead, would require a radius  $r_2$  depending on  $\theta$  caused by the fact that the critical counterflow velocity  $V_{c2}$  increases when one moves from the equator to the poles of the sphere. This means that  $r_2(\theta)$  has a maximum at the equator and minimum at the poles, that is the turbulent region is contained in an ellipsoid (see figure 2b). Note that in both figures 2 the region for  $r > r_2$  is filled of vortex array which suffers the perturbation coming from the inner turbulent region.

Because of the direct applications of these arguments on neutron stars, some other remark could be pointed out. It is to note that different forms of

the superfluid turbulent core have implication on the moment of inertia of the star which may influence the acceleration periods. These acceleration periods are a real topic which has deserved many interest for some researchers, but nothing has still been proved because the interior of the neutron stars are not directly observable. But, some more information on neutron stars can be obtained because Fig. 2 can be experimentally achieved: take a rotating sphere full of superfluid, which has a small heated sphere in its centre, then the details of the frontier between the vortex tangle and the array of parallel vortices (or of a highly polarized tangle) can be measured.

The procedure pointed out in Refs. [17] was generalized to derive an evolution equation for the superfluid velocity  $\mathbf{V}_s$ , thermodynamically consistent with the proposed evolution equation for  $L$ . The form (28) and (33) point out — in analogy with the conclusions reached in Ref. [17] but in a more direct way — that a tangle of a given  $L$  and  $\Omega$  has less resistance to the flow than a tangle with the same value of  $L$  but in absence of rotation. This fact may be attributed to the orientational influence of the rotation on the vortex lines.

In Ref. [17], the authors obtained the coupling force  $\mathbf{F}_{coupl}$ , between  $\mathbf{V}$  and  $\Omega$ , in the evolution equation for superfluid velocity coupled to the generalized Vinen equation. As pointed out along this paper, they distinguished essentially two cases, both corresponding to ours. Regarding the case of subsection 3.2 — in which the investigated term of the generalized Vinen equation (and alternative Vinen equation) depends on the angle between  $\mathbf{V}$  and  $\Omega$  —, in [17] the authors find  $\mathbf{F}_{coupl} \sim -L \cos(\hat{\mathbf{V}}\hat{\Omega})\Omega/\sqrt{\Omega}$  which differs from the corresponding (33) obtained in this paper for the dependence on  $V$ . Note also that the sign of coupling force (33) does not depend on the angle between  $\mathbf{V}$  and  $\Omega$  whereas that found by Jou and Mongiovì does. The coupling force (33) is, instead, similar to the last two terms of equation (4.12) of paper [17].

As regards the case of subsection 3.1 — in which the investigated term does not depend on the angle between  $\mathbf{V}$  and  $\Omega$  — it corresponds to case B of the paper [17], where the authors found  $\mathbf{F}_{coupl} \sim -L\sqrt{\Omega}\hat{\mathbf{V}}$  which differs from that proposed here. Again, the coupling force (28) is similar to the last two terms of equation (4.17) of paper [17].

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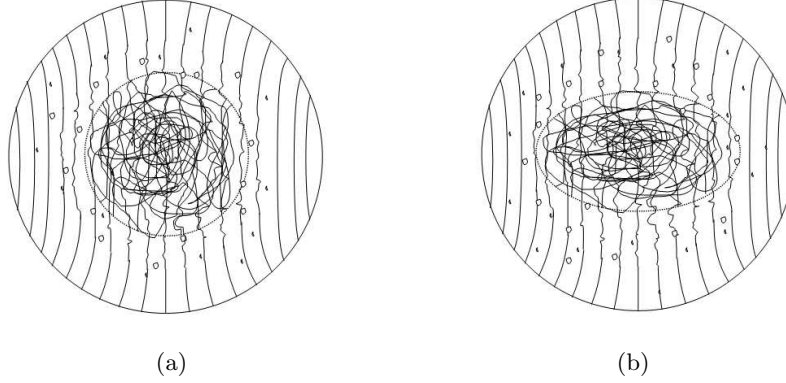


Figure 2: Sketch of a rotating sphere with an internal turbulent region and an external laminar region. The boundaries of the turbulent regions will be or not spherical depending on whether the critical heat flux (16) does not depend on the relative angle between heat flux and angular velocity or depends on it. The picture of the external region ( $r > r_2$ ) is only indicative, but a strongly polarized turbulent region could be more realistic.

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